PHY456 Lecture 5

More on Spin, Stern-Gerlach Apparatus

[Aside- Skipped in class] Tensor Products:

How is it that $\begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$ can be thought of as a tensor product, and that the norm is the sum of norms?

Consider two vector spaces a (n-dim) and b(2-dim)

$$a: \quad E_I = 1,2,\dots,N$$

$$b: \quad e_j = 1,2$$

Then $a \otimes b = E_I \otimes e_j$ is a 2n-dimensional basis.

Basis vectors satisfy orthogonality

$$E_I^\dagger E_J = \delta_{IJ},$$

$$e_i^\dagger e_j = \delta_{ij}$$

and $(E_J^\dagger \otimes e_j^\dagger) \cdot (E_I \otimes e_i) = \delta_{IJ}\,\delta_{ij}$.

For two arbitrary vectors in $a \otimes b$ space, we have that

$$V = V_{Ji}\, E_J \otimes e_i$$

$$u = u_{Ji}\, E_J \otimes e_i$$

and $$u^\dagger \cdot V = u^*_{Ji} V_{J'i'} \, (E_J \otimes e_i) \cdot (E_{J'} \otimes e_{i'})$$

$$= u^*_{Ji} V_{J'i'} \, \delta_{JJ'} \delta_{ii'}$$

$$= \sum_{J,i} u^*_{Ji} V_{Ji}$$

But how does this pertain to spin-$\frac{1}{2}$ Hilbert space?

b is 2-dimensional (index $i = \pm$)

a is $n \to \infty$ dimensional (index $J \to r$; continuous)

"position"

So $$V = V_{Ji}\, E_J \otimes e_i$$

$$= \left\{ \begin{matrix} V_+(\vec{r}) \\ V_-(\vec{r}) \end{matrix} \quad \text{for all } \vec{r} \right\}$$

and $$u^\dagger V = \sum_{J,i} u^*_{Ji} V_{Ji}$$

$$\rightarrow \quad = \int dr \left(u^*_+(r) V_+(r) + u^*_-(r) V_-(r) \right)$$

Stern-Gerlach Apparatus

→ Internalize rotations of spinors

Consider a beam of Hydrogen (ground-state) atoms

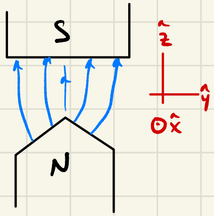
$$\psi_{initial}(r_{cm}, r) = \psi_y(r_{cm}) \begin{pmatrix} \psi_{100}(r) \\ 0 \end{pmatrix}$$

y: indicates we build a wave packet in y-direction

r_{cm}: cm of atom ≃ proton position

$\psi_{100} \equiv \psi_{n\ell m}(r)$ ground state

$\begin{pmatrix} \psi_{100} \\ 0 \end{pmatrix}$: means spin of e^- is up $\quad \hat{S}_z \begin{pmatrix} \psi_{100} \\ 0 \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} \psi_{100} \\ 0 \end{pmatrix}$



Magnet along B_z but is not constant along z, $\left(\frac{\partial B_z}{\partial z} < 0 \text{ by sharp edges} \right)$

→ H atom is neutral, not affected by Lorentz Force

but has magnetic moment due to e^-,

$$\hat{H}_B = -\frac{q}{m_e} \vec{B}(r_{cm}) \cdot \hat{\vec{S}}$$

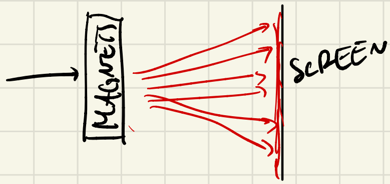
Classical magnetic moment is

$$E = -\vec{\mu} \cdot \vec{B}$$

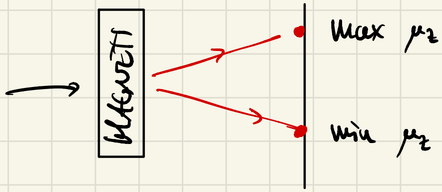
So force is $\vec{F} = -\nabla E = (\vec{\mu} \cdot \nabla)\vec{B}$

and $\vec{B} = B_z \hat{z}$ implies $\vec{F} = F_z \hat{z} = \mu_z \frac{\partial B}{\partial z} \hat{z}$.

For all kinds of $\vec{\mu}$, we expect a continuous image on a screen



but we find instead



Use Ehrenfests theorem, CM moves classically due to force

$$F_z = \mu_z \underbrace{\frac{\partial B_z}{\partial z}}_{<0} \quad , \quad \mu_z = \frac{q}{m}\langle S_z \rangle = -\frac{e}{m}\frac{\hbar}{2} < 0$$

So after magnet we have $\Psi^{(S_z\uparrow)}_{\text{"final"}}(r_{cm}, r) = \Psi_{g\uparrow}(r_{cm}) \begin{pmatrix} \psi_{100}(r) \\ 0 \end{pmatrix}$

wavepacket displaced up in $\hat{z}$ w/ some velocity in $\hat{z}$ ...

but if $\langle S_z \rangle_{\text{initial}} = -\frac{\hbar}{2}$, we'd have

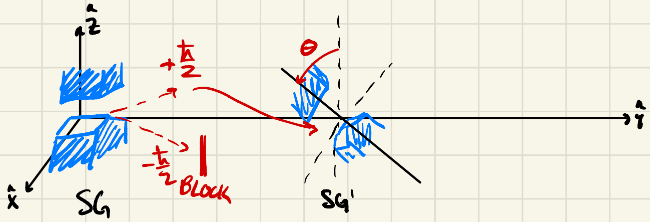
$$\Psi^{(S_z\downarrow)}_{\text{"final"}}(r_{cm}, r) = \Psi_{g\downarrow}(r_{cm}) \begin{pmatrix} 0 \\ \psi_{100}(r) \end{pmatrix}$$

In general: a superposition of initial up and down states.

$$\psi_{final}(r_{cm}, r) = \alpha\, \psi_{y+}(r_{cm}) \begin{pmatrix} \psi_{100}(r) \\ 0 \end{pmatrix} + \beta\, \psi_{y\downarrow}(r_{cm}) \begin{pmatrix} 0 \\ \psi_{100}(r) \end{pmatrix}$$

Thus after SG apparatus, spin of electron is correlated with position of atom in upper/lower beam.

What if we place another apparatus on the path of the $+\frac{\hbar}{2}$ upper beam, such that the second apparatus is oriented differently?



Only let $+S_z$ come from $SG_{(1)}$.

How many pass through SG' rotated at an angle θ in $x=z$ plane?

– project spins onto an axis $\vec{n}$, $\vec{n}\cdot\vec{S}$.

Then, eigenstates of $\vec{n}\cdot\hat{\vec{S}}$ = eigenstates of $\vec{n}\cdot\vec{\sigma}$.

Observe that eigenvalues of $\vec{n}\cdot\vec{\sigma}$ are ± 1,

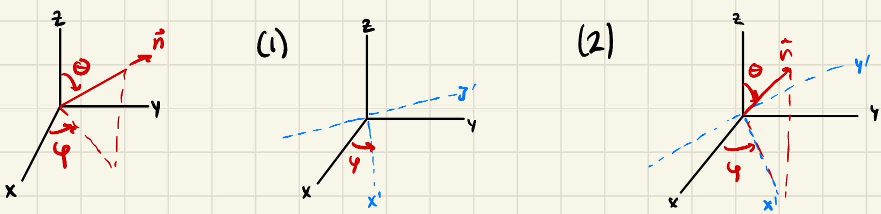
$$\left.\begin{array}{l} tr(\vec{n}\cdot\vec{\sigma}) = 0 \quad \text{and} \\ \det(\vec{n}\cdot\vec{\sigma}) = -1 \end{array}\right\} \Rightarrow \lambda = \pm 1.$$

We therefore want

$$(\vec{n}\cdot\vec{\sigma})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

We know that $\hat{z}\cdot\vec{\sigma}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\hat{z}\cdot\vec{\sigma}\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$.

Thus (1) rotate $\hat{z}$ on φ, then (2) around y' on θ:



$$\mathcal{U} = \mathcal{U}_1\mathcal{U}_2 = e^{-i\frac{\theta}{2}\hat{n}_{y'}\cdot\vec{\sigma}}\, e^{-i\frac{\varphi}{2}\sigma_z} = (*)$$

$$\hat{n}_{y'} = (-\sin\varphi, \cos\varphi, 0)$$

$$(*) = \left(\cos\tfrac{\theta}{2}\sigma_0 - i\sin\tfrac{\theta}{2}\,\hat{n}_{y'}\cdot\vec{\sigma}\right)\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2}\end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2}\sin\varphi - \sin\frac{\theta}{2}\cos\varphi \\ i\sin\frac{\theta}{2}\sin\varphi + \sin\frac{\theta}{2}\cos\varphi & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} e^{-i\varphi} \\ \sin\frac{\theta}{2} e^{i\varphi} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{\begin{matrix} \cos\frac{\theta}{2} e^{-i\varphi/2} \\ \sin\frac{\theta}{2} e^{i\varphi/2} \end{matrix}}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \underbrace{\begin{matrix} -\sin\frac{\theta}{2} e^{-i\varphi/2} \\ \cos\theta/2 \, e^{i\varphi/2} \end{matrix}}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \end{pmatrix}$$

eigenstate $\lambda = +1$ eigenstate $\lambda = -1$ (of $\vec{n}\cdot\vec{\sigma}$).

This is a good example of rotations.

Interlude: $u_B = e^{-i\varphi/2\,\sigma_z}$ and let $\varphi = 2\pi$.

Then $u_B = -\mathbb{1}$!

Because u_B is an $SU(2)$ matrix, not $SO(3)$.

There are x2 as many u as there are $SO(3)$ rotations. ($u, -u$ give same rotation, so does $\mathbb{1}, -\mathbb{1}$).

The spinor itself is not physical. What is the matrix element $\langle\varphi|\hat{O}|\psi\rangle$?

Spinors   observable

When both $\varphi \to -\varphi$ and $\psi \to -\psi$, $\langle\varphi|\hat{O}|\psi\rangle$ is unaffected.

Back to SG, SG'.

The SG' eigenstates along z' axis are

$$|z',+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$|z',-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.$$

Project onto S_z basis (incoming S_z spin up)

$$|z,+\rangle = \left(|z',+\rangle\langle z',+| + |z',-\rangle\langle z',-|\right)|z,+\rangle$$

$$= |z',+\rangle \left(\cos\tfrac{\theta}{2} \;\; \sin\tfrac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} + |z',-\rangle\left(-\sin\theta/2 \;\; \cos\tfrac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= |z',+\rangle\cos\tfrac{\theta}{2} + |z',-\rangle\left(-\sin\tfrac{\theta}{2}\right).$$

only these pass through SG' apparatus, fraction is $\cos^2\frac{\theta}{2}$.

$\theta = \pi$ (none pass)

$\theta = \frac{\pi}{2} \left(\frac{1}{2} \quad p\cos s \right)$